Singularities of sub-Riemannian exponential mappings, conjugate loci (caustics), wave fronts, cut loci and Carnot-Caratheodory small-balls (Recent results by Agrachev, El-Alaoui, Gauthier, Ge and Kupka)

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Singularities of sub-Riemannian exponential mappings, conjugate loci (caustics), wave fronts, cut loci and Carnot-Carathéodory small-balls (Recent results by Agrachev, El-Alaoui, Gauthier, Ge and Kupka).

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1 Introduction.

After reviewing fundamental notions of sub-Riemannian or nonholonomic or Carnot-Carathéodory (C-C) geometry, we shall explain the recent results [1][2][9], due to Aglachev, El-Alaoui, Gauthier, and Kupka, on singularities appearing in various geometric objects of generic sub-Riemannian or C-C metrics on $\mathbf{R}^3$ with the contact distribution. See also [10][11]. Also we compare these results with the previous results [22] by Vershik and Gershkovich on the left invariant sub-Riemannian metric of the 3-dimensional Heisenberg group.

One of extremely different features of sub-Riemannian geometry from Riemannian geometry appears in the fact that the closure of the conjugate locus as well as the cut locus of a point contains the original point, and, therefore, a C-C small-balls has singularities even if the radius is sufficiently small.

The geodesic flow for a sub-Riemannian metric naturally lives on the cotangent bundle, and it is reasonable to follow the Hamiltonian formalism [18]. In [1][2][9], in particular, using the classical Whitney's theorem on singularities of plane to plane mappings (with estimates), it has been investigated the **diffeomorphism type** of the germ at a point of the closure of the conjugate locus for a generic C-C metric on $\mathbf{R}^3$. However the method used there is limited to the three dimensional case.

To generalize the classification results of [1][2][9], to more higher dimensional cases, for instance to the Engel case on $\mathbf{R}^4$, it is natural, even in the three dimensional case, to apply **Lagrange and Legendre** (L-L) **singularity theory**, namely singularity theory for caustics and wave fronts [5], not the ordinary singularity theory of differentiable mappings, to sub-Riemannian geometry.

However we emphasize that our classification problem is local but **micro-global**; a global version of L-L singularity theory or L-L singularity theory at infinity is not fully investigated yet, as our fortune, (however see [12]), and therefore the application of singularity theory to sub-Riemannian geometry requires more improvement of L-L singularity theory itself.

There are other possibilities of applications of singularity theory to the problem of singularities of end-point mappings and abnormal geodesics can be found in [1][4], and to the problem of singularities of Pfaff systems and rigid curves [25].

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This short survey article is a revival of my talk given at RIMS in 29 January 1997. The subsequent progress on this subject can be seen in [3].

2 Sub-Riemannian geometry

Let M be a connected C^∞-manifold of dimension n, and D a C^∞-subbundle of the tangent bundle TM of M. We call D **non-holonomic** or **bracket generating** if, for each point $P \in M$, any $v \in T_PM$ is represented as a sum of iterated brackets of sections of D. In what follows we assume D is non-holonomic.

A **sub-Riemannian structure** g on (M, D) is a Riemannian metric on the non-holonomic subbundle D of TM; $g : D \oplus D \to \mathbf{R}$, positive definite symmetric bilinear form. We call the triplet (M, D, g) a **sub-Riemannian manifold.**

Example: Let

$$M = \mathbf{R}^3 = G = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \quad x, y, z \in \mathbf{R} \right\},$$

be the 3-dimensional Heisenberg group. In its Lie algebra

$$\mathcal{G} = T_1G = \left\{ \left(\begin{array}{ccc} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) \middle| \quad x, y, z \in \mathbf{R} \right\},$$

we set

$$X = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad Y = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right),$$

and $V = \langle X, Y \rangle_{\mathbf{R}}$. Then

$$[X, Y] = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) (=: Z).$$

Thus V defines a *left-invariant* non-holonomic subbundle D of TG of rank 2. Actually D is a *contact structure* on G defined by $dz - xdz = 0$. Moreover, if we give a metric on V, then we have a left-invariant Riemannian metric on D. We are going to study on generic perturbations of this left-invariant sub-Riemannian structure on $\mathbf{R}^3$.

Rashevsky-Chow's theorem says that, for any two points P, Q of M, there exists a piecewise differentiable path $c : [a, b] \to M$ such that $c(a) = P, c(b) = Q$ and that $\dot{c}(t) \in D_{c(t)}$, for almost every t. Paths satisfying the latter condition are called **admissible** or **horizontal.** The **length** of an admissible path c is defined by

$$L(c) = \int_a^b \|\dot{c}(t)\|_g dt.$$

Then the **Carnot-Carathéodory distance** is defined by

$$d(P,Q) = \text{C-C-}d(P,Q) = \inf\{L(c) \mid c \text{ is an admissible path from } P \text{ to } Q\}.$$

We set, for $x \in M$ and for $\varepsilon > 0$,

$$B_\varepsilon(P) = \{Q \in M \mid d(P,Q) < \varepsilon\}.$$

Fact (1): The metric C-C-d induces on M the original topology (as a manifold). In other words, $\{B_\varepsilon(P)\}_\varepsilon, \varepsilon > 0$, form a system of neighborhoods of P with respect to the manifold topology of M (cf. Ball-Box theorem [7]).

We call D **strongly bracket generating** (SBG) if, for each $P \in M$, and for a section X of D with $X(P) \neq 0$, any $v \in T_P M$ is represented as a sum of a section of D and a single bracket of X and a section of D.

Fact (2): If D is SBG, e.g. contact, then, for a sufficiently small $\varepsilon > 0$, $B_\varepsilon(P)$ is homeomorphic to the Euclidean ball, and the closure

$$\bar{B}_\varepsilon(P) = \{Q \in M \mid d(P,Q) \leq \varepsilon\}$$

is homeomorphic to the Euclidean closed ball. However $\bar{B}_\varepsilon(P)$ $(P \in M, 0 < \varepsilon << 1)$, has always *singularities* with respected to the differentiable structure of M; there exists a point Q on the boundary of $\bar{B}_\varepsilon(P)$ such that the relative germ $(M, \bar{B}_\varepsilon(P), Q)$ at Q is homeomorphic but not diffeomorphic to $(\mathbf{R}^n, \{x_n \geq 0\}, 0)$.

An admissible path $c : [a,b] \to M$ is called a **minimizer** with respect to the C-C distance, if $L(c) = d(c(a), c(b))$. An admissible path $c : [a,b] \to M$ is called a **local**

minimizer if, for any $t_0 \in [a,b]$, there exists a closed interval $[\alpha,\beta]$ containing t_0 as an interior point in $[a,b]$ such that $c|_{[\alpha,\beta]}$ is a minimizer.

It is known that a local minimizer is necessarily an **extremal**: Extremals are divided into **normal extremals** and **abnormal extremals**. The notion of normal extremals, which we will explain below, belongs to sub-Riemannian geometry; while the notion of abnormal extremals is in non-holonomic geometry, that is independent of sub-Riemannian structure g. Abnormal extremals live in $D^\perp \subset T^*M$ [18].

Fact (3): If D is SBG, e.g. contact, then there exists no non-constant abnormal extremal. Moreover if $P, Q \in M$ are sufficiently near, then there exists a normal extremal such that $L(c) = d(P,Q)$.

Fix $P \in M$. Take local frame $X_1, \dots, X_r$ of D over a neighborhood of P. Then a sub-Riemannian structure on (M, D) near P is uniquely determined such that $X_1, \dots, X_r$ are orthonormal.

Define the **sub-Riemannian Hamiltonian** $h : T^*M \to \mathbf{R}$ by

$$h(\xi) = -\frac{1}{2}(\langle \xi, X_1\rangle^2 + \dots + \langle \xi, X_r\rangle^2),$$

for $\xi \in T^*M$. Here $\langle\ \cdot\ ,\ \cdot\ \rangle : T^*M \oplus TM \to \mathbf{R}$ denotes the natural pairing. Then we see that h is critical just along $h^{-1}(0) = D^\perp \subset T^*M$. Moreover normal extremals are projections of solutions of the Hamiltonian flow defined by the Hamiltonian h.

To analyze sub-Riemannian structure through the Hamiltonian, we review in the next section on the Hamiltonian formalism.

3 Hamiltonian formalism

Let M be a C^∞ manifold of dimension n, $h : T^*M \to \mathbf{R}$ a C^∞ function. We assume h is homogeneous of degree m with respect to the fiber coordinates of $\pi : T^*M \to M$. (For the sub-Riemannian Hamiltonian in the previous section, we see $m = 2$.)

We denote by $\theta = \theta_M$ the Liouville 1-form on T^*M, and by $\omega = d\theta$ the symplectic 2-form on T^*M. For a local coordinates $q_1, \ldots, q_n$ of M, and for the coresponding fiber coordinates $p_1, \ldots, p_n$, we have $\theta = \sum p_i dq_i$ and $\omega = \sum dp_i \wedge dq_i$. Then the Hamiltonian vector field $\vec{h}$ on T^*M with Hamiltonian h is defined by

$$\vec{h} \rfloor \omega = -dh.$$

Locally

$$\vec{h} = \sum h_{q_i} \frac{\partial}{\partial p_i} - h_{p_i} \frac{\partial}{\partial q_i}.$$

We see that

$$\langle \theta, \vec{h} \rangle = -\sum p_i h_{p_i} = -mh.$$

In other words, $\vec{h} \rfloor \theta = -mh$.

Let $E = \sum p_i \frac{\partial}{\partial p_i}$ denote the *Euler field* over T^*M. Then $Eh = mh$. If $h(P) \neq 0$, then $dh(P) \neq 0$. Therefore the set of critical points of h is contained in $h^{-1}(0)$. In particular, for $c \neq 0$, the level hypersurface $S = h^{-1}(c)$ is non-singular. Also we see that $E \rfloor \omega = \theta$, namely E is a *Liouville field*, and therefore, denoting by L the Lie derivative, we have

$$L_E \omega = E \rfloor d\omega + d(E \rfloor \omega) = d\theta = \omega.$$

Then we see (cf. [14][13]):

Lemma 3.1 *$\theta|_S$ is a contact form on $S = h^{-1}(c)$, $c \neq 0$, and $\vec{h}|_S$ is a contact vector field. In fact more strictly we see $L_{\vec{h}}(\theta|_S) = 0$.*

Proof: We have

$$\theta \wedge (d\theta)^{n-1} = \theta \wedge \omega^{n-1} = (E \rfloor \omega) \wedge \omega^{n-1} = \frac{1}{n} E \rfloor \omega^n \neq 0,$$

on S. Therefore $\theta|_S$ is a contact form. Moreover $\vec{h}$ is tangent to S, and

$$L_{\vec{h}} \theta = \vec{h} \rfloor \omega + d(\vec{h} \rfloor \theta) = -dh - mdh = -(m+1)dh = 0,$$

on S. $\square$

4 Sub-Riemannian wavefronts.

Now we return to the sub-Riemannian geometry.

By Lemma 3.1, $S = h^{-1}(-\frac{1}{2})$ is a contact manifold with the contact form $\theta|_S$. Denote by Φ_t the contact flow on S defined by $\vec{h}$. The constant $c = -\frac{1}{2}$ is chosen so that the time parameter of solution curves (normal extremals), coincide with their C-C arc-lengths. Remark that Φ_t is well-deined for sufficiently small t.

Set $C = S \cap T_P^*M \cong S^{r-1} \times \mathbf{R}^{n-r}$. Then $\theta|_C = 0$ and therefore C is a *Legendre submanifold* of S. Consider the transform $\Phi_t(C) \subset S$ and its projection $W_t = \pi(\Phi_t(C)) \subset M$ by the bundle projection $\pi : T^*M \to M$. We call W_t the **wavefront** from P of time t.

Then, by Fact (3), we observe

Lemma 4.1 *If D is SBG, and $P \in M$, then*

$$\bar{B}_\varepsilon(P) = \{Q \in M \mid d(P,Q) \leq \varepsilon\} = \bigcup_{0 \leq t \leq \varepsilon} W_t.$$

Our fundamental problem is: How singular are W_t, $\bar{B}_\varepsilon$? For the study on singularities of $\bar{B}_\varepsilon(P)$, first we have to investigate the singularities of W_t.

Define the **exponential map** $e : \mathbf{R}_+ \times C \to M$ near $0 \times C$ by $e(t,\xi) = \pi(\Phi_t(\xi))$.

For $\xi \in C$, denote by $\tau(\xi)$ the **escape time**, that is the time that $\pi(\Phi_t(\xi))$ goes out the fixed neighborhood of P. Then set

$$\begin{aligned} t_c(\xi) \;&=\; \sup\{t \in \mathbf{R}_+ \mid 0 < t < \tau(\xi);\ 0 < t' < t \Rightarrow \\ &\qquad\qquad e_* : T_{(t',\xi)}(\mathbf{R}_+ \times C) \to T_{e(t',\xi)}M \text{ is isomorphic } \}, \end{aligned}$$

the **first conjugate time**.

Lemma 4.2 *$\Phi : \mathbf{R}_+ \times C \to T^*M$, $\Phi(t,\xi) = \Phi_t(\xi)$, is a* **Lagrange immersion**.

Thus the exponential map e is a Lagrange map. The singular locus of e coincides with the trace of singular points of wavefronts.

Proof: It suffices to show that $\vec{h}$ does not tangent to C anywhere. Recall $Eh = -2h$, so, on $T^*M - \{h = 0\}$, $h_{p_i} \neq 0$, for some i. Therefore $\vec{h}$ does not tangent to $T^*_P M$ along $\{h \neq 0\}$. □

Now let $M = \mathbf{R}^3$ and $D \subset TM$ be a contact distribution. Let $P \in \mathbf{R}^3$. Take a local frame X, Y of D. Then recall that

$$h(\xi) = -\frac{1}{2}(\langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2).$$

We take the coordinates of $C \cong S^1 \times \mathbf{R}$, cylinder, as follows: Choose the 1-form α satisfying (1) $\ker \alpha = D$, and (2) $d\alpha(X, Y) = 1$. Take the unique vector field ζ on M such that $\zeta \rfloor (\alpha \wedge d\alpha) = d\alpha$. Define a basis $\alpha_1, \alpha_2, \alpha_3$ of $T^*_P M$ by

$$\begin{array}{lll} \langle \alpha_1, X(P)\rangle = 1, & \langle \alpha_1, Y(P)\rangle = 0, & \langle \alpha_1, \zeta(P)\rangle = 0, \\ \langle \alpha_2, X(P)\rangle = 0, & \langle \alpha_2, Y(P)\rangle = 1, & \langle \alpha_2, \zeta(P)\rangle = 0, \\ \langle \alpha_3, X(P)\rangle = 0, & \langle \alpha_3, Y(P)\rangle = 0, & \langle \alpha_3, \zeta(P)\rangle = \langle \alpha, \zeta\rangle(0). \end{array}$$

Then we define the cylindrical coordinates $T^*_P M - \{h = 0\} \cong \mathbf{R}^3 - \{(0,0)\} \times \mathbf{R}$ by

$$\xi = R\cos\varphi\alpha_1 + R\sin\varphi\alpha_2 + r\alpha_3,$$

where $0 \leq R,\ 0 \leq \varphi < 2\pi,\ r \in \mathbf{R}$. Then

$$C = \{\xi \in T^*_P M \mid \langle \xi, X\rangle^2 + \langle \xi, Y\rangle^2 = 1\} = \{\xi \in T^*_P M \mid R = 1\},$$

which is parametrized by φ and r. Thus we have $C \cong S^1 \times \mathbf{R}$.

Then the main result is the following:

Theorem 4.3 ([1][2][9]) *Fix X, Y and $P \in M = \mathbf{R}^3$. Then there exist $a \in \mathbf{R}$ and $b \in \mathbf{R}_+$ such that, setting $\rho = 1/r$,*

$$t_c(\varphi, \rho) = 2\pi\rho + a\rho^3 + O(\rho^4), \quad (\rho > 0).$$

We define $q_c : C \to M$ *by* $q_c(\xi) = e(t_c(\xi), \xi)$. *Then moreover there exists a system of cordinates of* M *near* P *such that*

$$q_c(\varphi, \rho) = \pi\rho^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b\rho^3 \begin{pmatrix} \cos^3\varphi \\ -\sin^3\varphi \\ 0 \end{pmatrix} + O(\rho^4).$$

The image $q_c(C) \subset M$ is called the **first conjugate locus** or the **caustic**. Using the classical Whitney's theorem it is shown in [1][2][9] that the caustic is diffeomorphic to a cone of the *asteroid.*

5 Figures

Figures 1 and **2** are taken from [22]: Figure 1 is a very rough picture of the wavefront for the Heisenberg case. The more detailed one is presented in Figure 2.

Figures 3, 4, 5 show several parts of the Heisenberg wavefront, which are drawn by **Mathematica.**

Figure 6 is from [7], which shows the C-C small balls for the Heisenberg case.

The zoomed-out picture of a generic sub-Riemannian wavefront is presented in **Figure 7**, taken from [2].

Figures 8 and **9** are zoomed-in picture: There exists a curve γ in $M = \mathbf{R}^3$ such that, for $P \in M - \gamma$, each conical point of the Heisenberg wavefront is perturbed into 4 *swallowtails*, while, for $P \in \gamma$, into 6 swallowtails.

Figure 10 and **11** are hand-written pictures: Figure 10 describes the ways of perturbations of conical singularities of the Heisenberg wavefront to a generic one. Figure 11 shows the singuarities of C-C small balls.

Sub-Riemannian caustics in the Heisenberg case and in generic case are given in **Figure 12**: The latter figure is taken from [5].

Figure 13 is from [2], which shows the half part of generic caustic, for $P \in M - \gamma$, and, for $P \in \gamma$, respectively.

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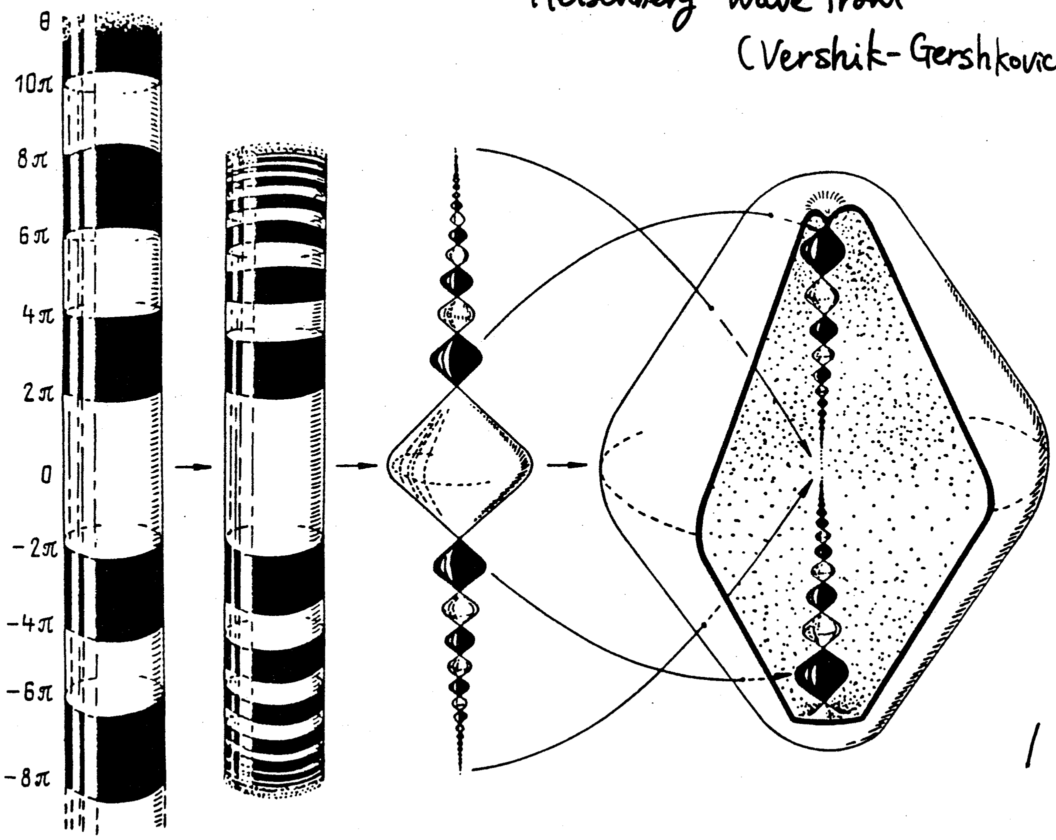
Heisenberg Wave front
(Vershik-Gershkovich)



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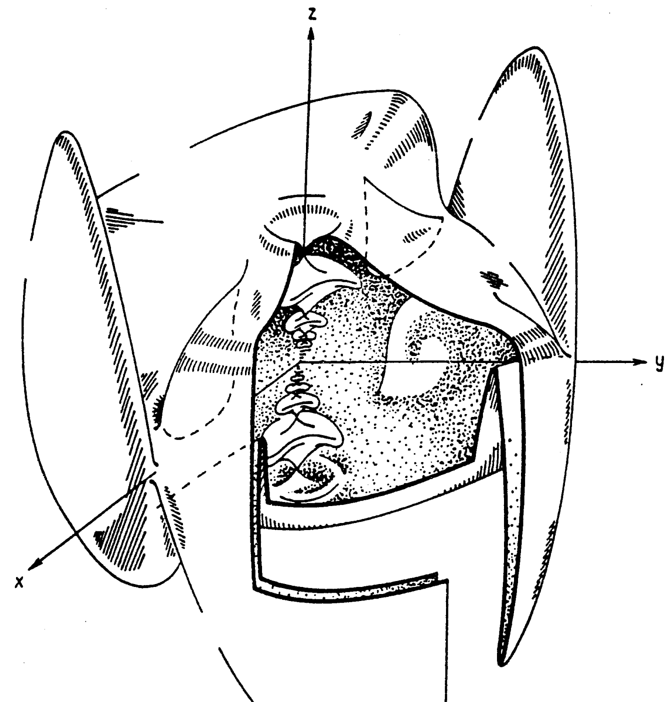

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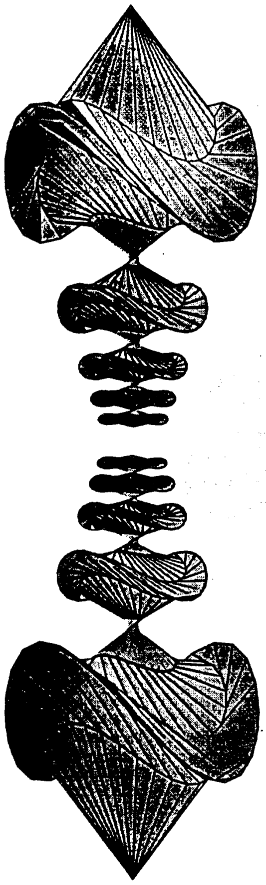

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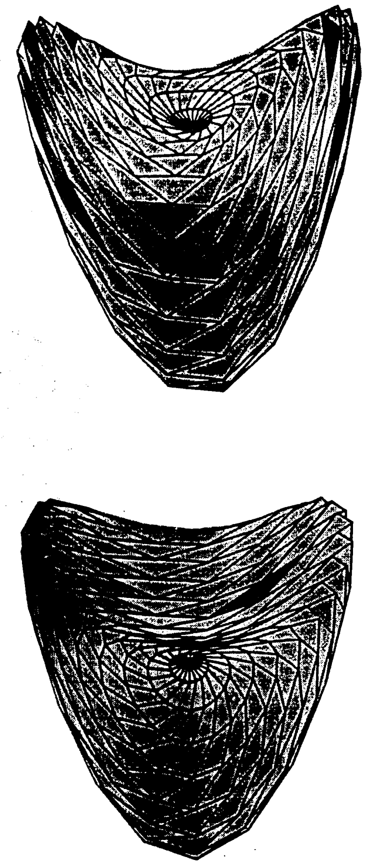

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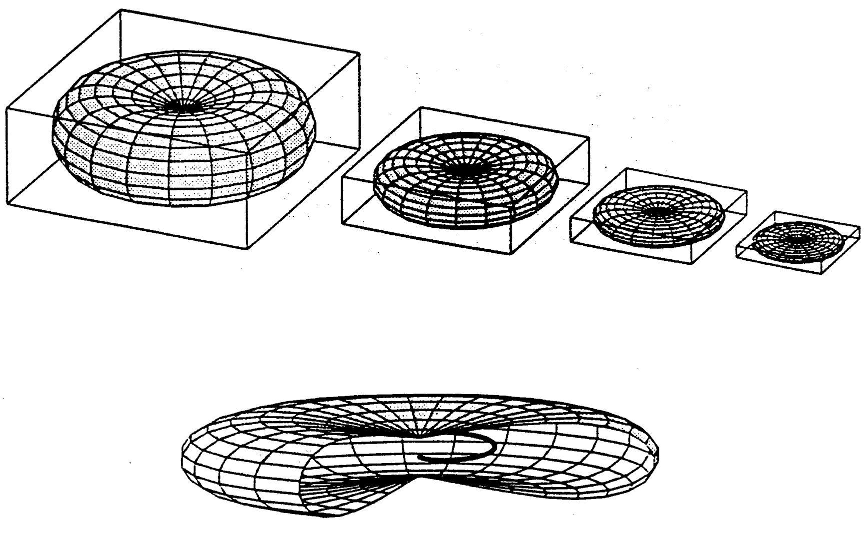
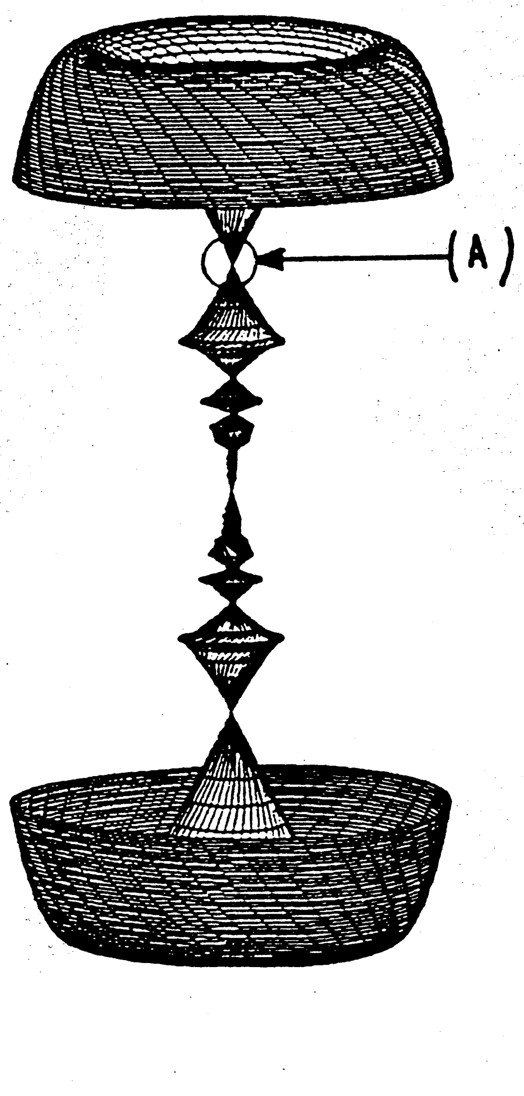


from Bellaïche 6

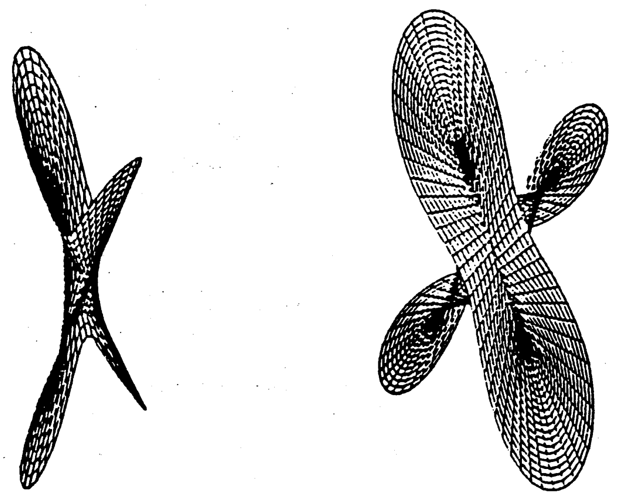
generic Sub-Riemannian wave front



7

from AAGK

generic Sub-Riemannian wave front



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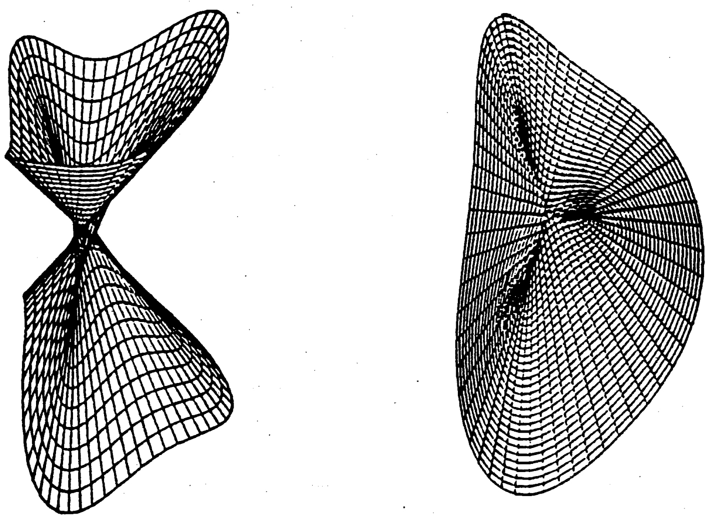
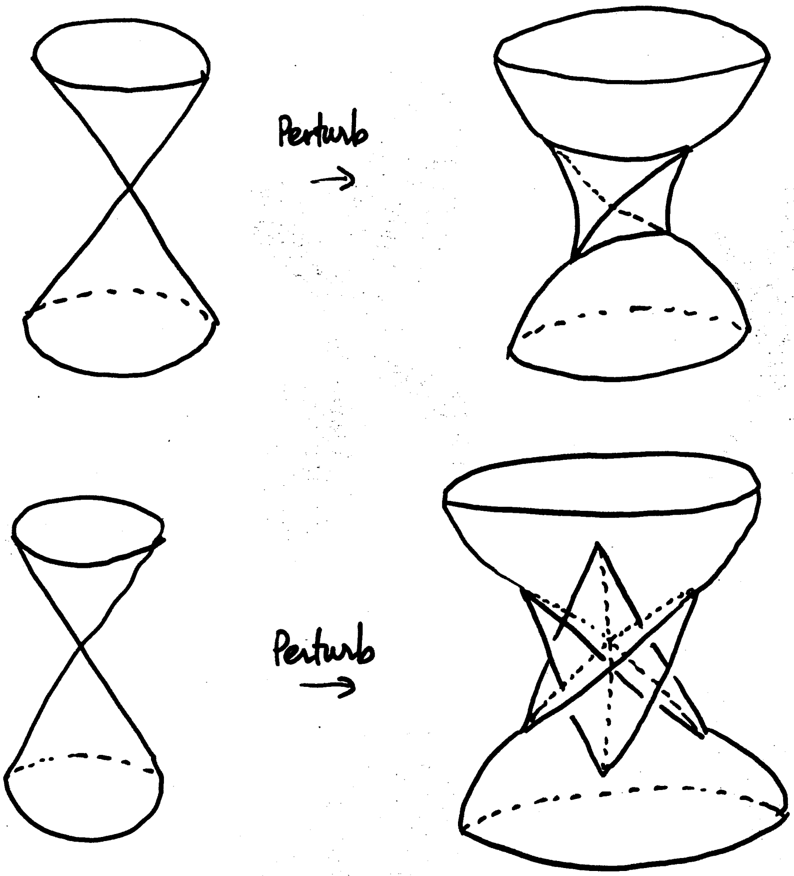


9

from AAGK

Sub-Riemannian perturbation of a cone (of a wavefront)

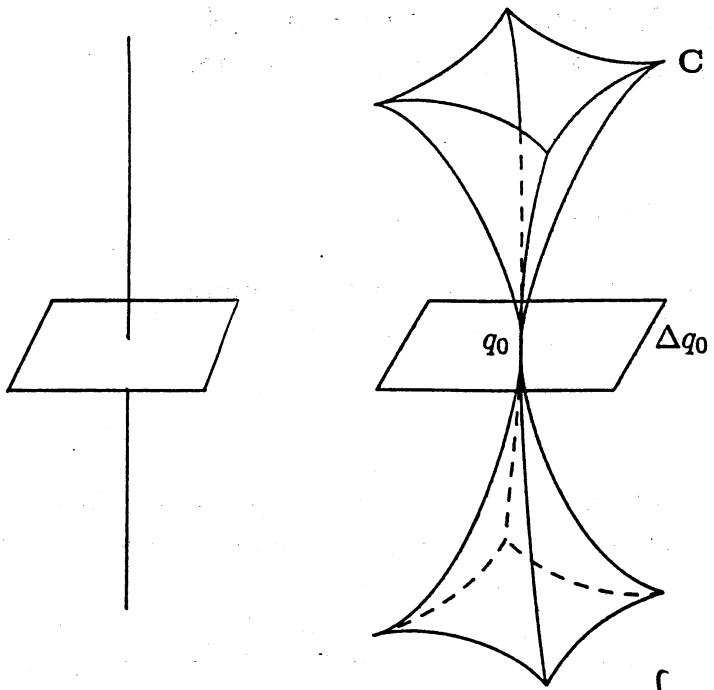


10

Generic C-C small ball

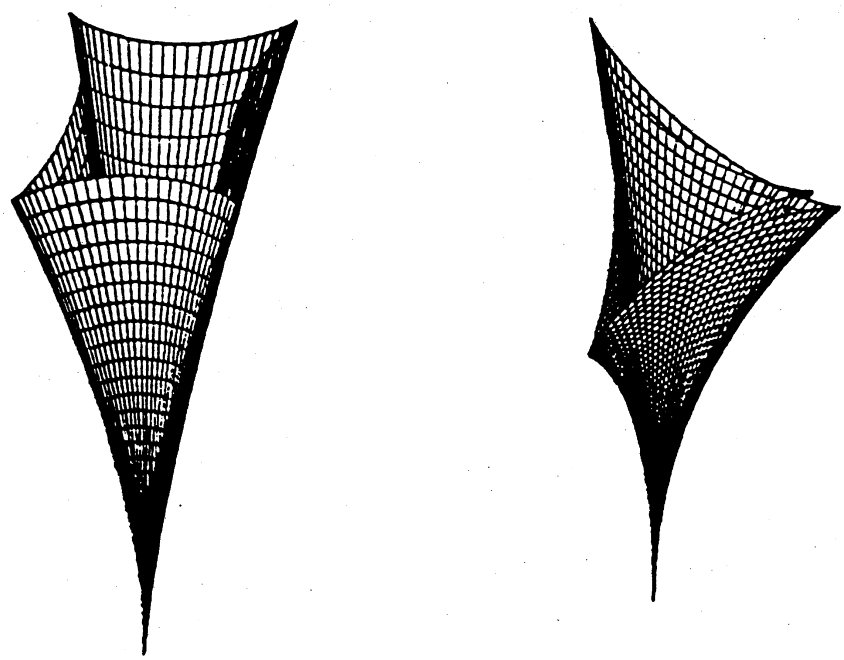
11

Generic sub-Riemannian Caustic.



12

from Agrachev.



from AAGK 13